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Covariant approximation schemes for calculation of the heat kernel in quantum field theory

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This paper is an overview on our recent results in the calculation of the heat kernel in quantum field theory and quantum gravity. We introduce a deformation of the background fields (including the metric of a curved spacetime manifold) and study various asymptotic expansions of the heat kernel diagonal associated with this deformation. Especial attention is payed to the low-energy approximation corresponding to the strong slowly varying background fields. We develop a new covariant purely algebraic approach for calculating the heat kernel diagonal in low-energy approximation by taking into account a finite number of low-order covariant derivatives of the background fields, and neglecting all covariant derivatives of higher orders. Then there exist a set of covariant differential operators that together with the background fields and their low-order derivatives generate a finite dimensional Lie algebra. In the zeroth order of the low-energy perturbation theory, determined by covariantly constant background, we use this algebraic structure to present the heat operator in the form of an average over the corresponding Lie group. This simplifies considerably the calculations and allows to obtain closed explicitly covariant formulas for the heat kernel diagonal. These formulas serve as the generating functions for the whole sequence of the Hadamard-Minakshisundaram- De Witt-Seeley coefficients in the low-energy approximation.

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1. Introduction

The heat kernel for an elliptic differential operator H acting on sections of a vector bundle over a manifold M plays a very important role in various areas of mathematical physics, especially in quantum field theory and quantum gravity [1-12]. It is defined as the kernel of the one-parameter semigroup (or heat operator), $U(t) = \exp(-tH)$, viz.

$$U(t|x, x') = \exp(-tH)\mathcal{P}(x, x')g^{-1/2}\delta(x, x'), \tag{1.1}$$

where $\mathcal{P}(x, x')$ is the parallel displacement operator of quantum fields $\varphi(x)$ (sections of the vector bundle) from the point x to the point x' along the geodesic.

The heat kernel determines among others such fundamental objects of the quantum field theory as the Green function, the kernel of the resolvent, $(H+\lambda)^{-1}$, the zeta-function, [13]

$$\zeta(p) = \mathrm{Tr}\, H^{-p} = \frac{1}{\Gamma(p)} \int\limits_0^\infty dt t^{p-1} \mathrm{Tr}\, U(t), \tag{1.2}$$

the functional determinant, $\mathrm{Det}\, H$, and, hence, the one-loop effective action

$$\Gamma_{(1)} = \tfrac{1}{2} \log \mathrm{Det}\, H = -\tfrac{1}{2}\zeta'(0). \tag{1.3}$$

The functional trace 'Tr' in (1.2) is defined according to

$$\mathrm{Tr} U(t) = \int\limits_M dx g^{1/2} \mathrm{tr}[U(t)], \tag{1.4}$$

where 'tr' is the usual matrix trace and

$$[U(t)] = U(t|x, x) \tag{1.5}$$

is the heat kernel in coinciding points, so-called heat kernel diagonal.

In quantum field theory, the manifold M is, usually, taken to be a d-dimensional Riemannian manifold with a metric, $g_{\mu\nu}$, of Euclidean (positive) signature. The most important operators are the second order elliptic operators of Laplace type

$$H = -\Box + Q, \tag{1.6}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the Laplacian, ∇_μ is a connection on the vector bundle and Q is an endomorphism of this bundle. In other words the operator H acts on quantum fields $\varphi(x)$, $Q(x)$ is a matrix valued potential term, ∇_μ is the covariant derivative defined with a connection, a Yang-Mills gauge field, $\mathcal{A}_\mu$. The gauge field strength (Yang-Mills curvature), $\mathcal{R}_{\mu\nu}$, is given by the commutator of covariant derivatives

$$[\nabla_\mu, \nabla_\nu]\varphi = \mathcal{R}_{\mu\nu}\varphi. \tag{1.7}$$

Obviously, the heat kernel is calculable *exactly* only in exceptional cases of background fields configurations, (see, for example [14]). On the other hand, to get the quantum amplitudes one has to calculate the effective action as the functional of background fields of *general* type. That is why one needs to develop *approximate* methods for calculation of the heat kernel in *general* case.

In quantum gravity and gauge theories the effective action is a *covariant* functional, i.e. it is invariant under diffeomorphisms and local gauge transformations. That is why the approximations for calculating the effective action must be *manifestly covariant*, i.e. they have to preserve the general covariance at *each order*.

Except for the well-known Schwinger-De Witt expansion [1-6] there are two covariant approximation schemes available [3]: *i)* the high-energy one, which corresponds to weak rapidly varying background fields (short waves), and *ii)* the low-energy approximation corresponding to the strong slowly varying background fields (long waves). The high-energy approximation was studied in [5,6,15-17] where the heat kernel and the effective action in second [5,6,15,16] and third [17] order in background fields (curvatures) were calculated. The low-energy approximation in various settings was studied in [18-21]. The authors of these papers summed up some particular terms in the heat kernel asymptotic expansion, such as the scalar curvature terms [18,19] or terms without derivatives of the potential term [20,21] etc.

In our recent papers [22-27] we studied the low-energy approximation in quantum gravity and gauge theories and developed a new purely algebraic *covariant* approach for calculating the heat kernel near diagonal. The point is that in low-energy approximation the covariant derivatives of the curvatures and the potential term (but not the curvature and the potential term themselves!) are small. Therefore, one can treat them perturbatively, the zeroth order of this perturbation theory corresponding to the covariantly constant background fields.

In particular, the following cases were considered:

i) covariantly constant gauge field strength and the potential term in flat space, $\nabla_\mu \mathcal{R}_{\alpha\beta} = \nabla_\mu Q = R_{\alpha\beta\gamma\delta} = 0$, [22,23,26],

ii) covariantly constant Riemann curvature and the potential term without the Yang-Mills curvature, $\nabla_\mu R_{\alpha\beta\gamma\delta} = \nabla_\mu Q = \mathcal{R}_{\mu\nu} = 0$, [22, 24-26],

iii) covariantly constant Yang-Mills curvature and the potential term with nonvanishing first and second derivatives in flat space, $\nabla_\mu \mathcal{R}_{\alpha\beta} = \nabla_\mu \nabla_\nu \nabla_\lambda Q = R_{\alpha\beta\gamma\delta} = 0$ [27]. In the [28] this method was applied for the calculation of the effective potential and the investigation of the vacuum structure of non-Abelian gauge theories.

One should stress from the very beginning that our analysis is *purely local.* We are not interested in the influence of topology but concentrate our attention rather on local effects. Of course, there are always special global effects (Casimir like effects, influence of boundaries, presence of closed geodesics etc.) that do not show up in the local study of the heat kernel. However, our aim is to investigate only those general physical situations where the contribution of these effects is small in comparison with local part. We are not going to present some exact result for specific background fields, but to develop some general approximation schemes of calculations. The algebraic approach elaborated in our papers [22-27] should be thought as a framework for a perturbation theory in non-homogeneity.

2. Asymptotic expansions

Let us call the Riemann curvature tensor $R_{\mu\nu\alpha\beta}$, the Yang-Mills curvature $\mathcal{R}_{\mu\nu}$ and the potential term Q the *background curvatures* or simply curvatures and denote them symbolic by $\Re = \{R_{\mu\nu\alpha\beta}, \mathcal{R}_{\mu\nu}, Q\}$. Let us introduce, in addition, the *infinite* set of all covariant derivatives of the curvatures, so-called *background jets*,

$$\mathcal{J} = \{\Re_{(i)}\}, \qquad \Re_{(i)} = \{\nabla_{\mu_1} \cdots \nabla_{\mu_i} \Re\}. \tag{2.1}$$

The whole set of the jets $\mathcal{J}$ completely describes the background, at least locally.

Let us make a deformation of the background fields by introducing some deformation parameters α and ε

$$g_{\mu\nu} \to g_{\mu\nu}(\alpha, \varepsilon), \qquad \mathcal{A}_\mu \to \mathcal{A}_\mu(\alpha, \varepsilon), \qquad Q \to Q(\alpha, \varepsilon) \tag{2.2}$$

in such a way that the jets transform uniformly

$$\Re_{(i)} \to \alpha\varepsilon^i \Re_{(i)}. \tag{2.3}$$

Such deformation of the background fields lead to the corresponding deformation of the operator H (1.6)

$$H \to H(\alpha, \varepsilon) \tag{2.4}$$

and the heat kernel

$$U(t) \to U(t; \alpha, \varepsilon). \tag{2.5}$$

Let us note that because of the transformation law (2.3) this deformation is *manifestly covariant*. Therefore, it gives a natural framework to develop various approximation schemes based on asymptotic expansions of the heat kernel in the the deformation parameters. It is obvious that the limit $\alpha \to 0$ corresponds to the small curvatures, $\Re \to 0$, i.e. to the covariant perturbation theory [17], and the other limit, $\varepsilon \to 0$, corresponds to small covariant derivatives of the curvatures, $\nabla_\mu \Re \to 0$, i.e. to the long-wave approximation [22-27]. More precisely, we recognize two cases:
i) the high-energy approximation,

$$\nabla\nabla\Re \gg \Re\Re \qquad \text{or} \qquad \varepsilon^2 \gg \alpha,$$

and ii) the low-energy approximation,

$$\nabla\nabla\Re \ll \Re\Re \qquad \text{or} \qquad \varepsilon^2 \ll \alpha.$$

2.1. Schwinger - De Witt asymptotic expansion

First of all, there is an asymptotic expansion of the heat kernel as $t \to 0$ (Schwinger - De Witt expansion) [2,5-10]

$$[U(t; \alpha, \varepsilon)] \sim (4\pi t)^{-d/2} \sum_{k \geq 0} \frac{(-t)^k}{k!} a_k(\alpha, \varepsilon). \tag{2.6}$$

This expansion is purely local and does not depend, in fact, on the global structure of the manifold. Its famous coefficients a_k, Hadamard - Minakshisundaram - De Witt - Seeley (HMDS) coefficients, are local invariants built from the background curvatures and their covariant derivatives [1-12,29-33]. The HMDS-coefficients play a very important role both in physics and mathematics and are closely connected with various sections of mathematical physics such as spectral geometry, index theorem, trace anomalies, Korteweg - de Vries hierarchy etc. [7,12,33].

One can classify all the terms in a_k according to the number of curvatures and their derivatives. First, there are terms linear in the curvature, then it follows the class of terms quadratic in the curvature, etc.. The last class of terms does not contain any covariant derivatives at all but only the powers of the curvatures. This general structure emerges by the expansion of a_k in the deformation parameters

$$a_k(\alpha, \varepsilon) = \sum_{0 \le n \le k} \alpha^n \varepsilon^{2k-2n} a_{k,n}. \tag{2.7}$$

Here $a_{k,n}$ are the homogeneous parts of a_k of order n in the curvatures that can be symbolically written in the form

$$a_{k,n} = \sum_{\substack{i_1,\dots,i_n \ge 0 \\ i_1+\cdots+i_n = k-2n}} \sum \Re_{(i_1)} \cdots \Re_{(i_n)}, \tag{2.8}$$

where the second summation is over different invariant structures. The first coefficient reads simply

$$a_0 = 1, \tag{2.9}$$

and the higher order coefficients a_k, $(k \ge 1)$ have the following homogeneous parts [5,6]

$$\begin{aligned}
a_{k,0} &= 0, \\
a_{k,1} &= -\alpha_k^{(1)} \Box^{k-1} Q + \alpha_k^{(1)} \Box^{k-1} R, \\
a_{k,2} &= \beta_k^{(1)} Q \Box^{k-2} Q + 2\beta_k^{(2)} \mathcal{R}_{\alpha\mu} \nabla^\alpha \Box^{k-3} \nabla_\nu \mathcal{R}^{\nu\mu} - 2\beta_k^{(3)} Q \Box^{k-2} R \\
&\quad + \beta_k^{(4)} R_{\mu\nu} \Box^{k-2} R^{\mu\nu} + \beta_k^{(5)} R \Box^{k-2} R + \nabla \left(\sum_{0 \le i \le 2k-3} \sum \nabla^i \Re \nabla^{2k-3-i} \Re \right), \\
&\dots \\
a_{k,k} &= \sum \Re^k,
\end{aligned} \tag{2.10}$$

where $\alpha_k^{(i)}$ and $\beta_k^{(i)}$ are numerical constants. Note that altogether there are only five quadratic invariant structures (up to the total derivatives) but very many structures of the type $\Re^k$.

The first coefficients, a_0, a_1, a_2, were calculated a long time ago by De Witt [2], a_3 was calculated by Gilkey [29] and the next coefficient, a_4, was calculated for the first time in general case in our PhD thesis [5] and published in [6,30,31] and in the case of scalar operators in [32]. The linear and quadratic parts in the HMDS-coefficients, i.e. $a_{k,1}$ and

$a_{k,2}, (k \geq 2)$, were also calculated in our PhD thesis [5] and published in [6,15,16]. The quadratic part was calculated only up to a total derivative. The same results were obtained completely independent in [34]. The next cubic order in curvature, $a_{k,3}$ was studied in [17]. The terms without the derivatives, $a_{k,k}$, in general case are unknown. The calculation of these terms is an open and very interesting and important problem.

2.2. High-energy asymptotic expansion

Let us consider now the asymptotic expansion in the limit $\alpha \to 0$ of the perturbation theory. One can show that it has the form

$$[U(t;\alpha,\varepsilon)] \sim (4\pi t)^{-d/2} \sum_{n\geq 0} (\alpha t)^n h_n(t;\varepsilon), \tag{2.11}$$

where $h_n(t,\varepsilon)$ are some *nonlocal* functionals that have the following asymptotic expansion as $t \to 0$

$$h_n(t;\varepsilon) \sim \sum_{l\geq 0} \frac{(-1)^{n+l}}{(n+l)!} (\varepsilon^2 t)^l a_{n+l,n}. \tag{2.12}$$

The first functionals h_n are [5,6,15,16]

$$\begin{aligned}
h_0(t;\varepsilon) &= 1, \\
h_1(t;\varepsilon) &= t\left\{F_1(\varepsilon^2 t\,\Box)Q - F_2(\varepsilon^2 t\,\Box)R\right\}, \qquad (2.13)\\
h_2(t;\varepsilon) &= \frac{t^2}{2}\Big\{QF_{(1)}(\varepsilon^2 t\,\Box)Q + 2\mathcal{R}_{\alpha\mu}\nabla^\alpha \frac{1}{\Box} F_{(3)}(\varepsilon^2 t\,\Box)\nabla_\nu \mathcal{R}^{\nu\mu} - 2QF_{(2)}(\varepsilon^2 t\,\Box)R \\
&\qquad + R_{\mu\nu}F_{(4)}(\varepsilon^2 t\,\Box)R^{\mu\nu} + RF_{(5)}(\varepsilon^2 t\,\Box)R\Big\} + \text{total derivative},
\end{aligned}$$

where $F_{(i)}(z)$ are the formfactors, i.e. some analytic functions. One can show that the formfactors $F_{(i)}(z)$ are *entire* functions, i.e. they are analytic in the whole complex plane. The explicit form of these functions was obtained in our PhD thesis [5] and published in the papers [6,15,16]. The *third* order in curvatures of the covariant perturbation theory was investigated in [17].

2.3. Low-energy asymptotic expansion

The low-energy approximation corresponds to the asymptotic expansion of the deformed heat kernel as $\varepsilon \to 0$

$$[U(t;\alpha,\varepsilon)] \sim (4\pi t)^{-d/2} \sum_{l\geq 0} (\varepsilon^2 t)^l u_l(t;\alpha). \tag{2.14}$$

The coefficients u_l are essentially *non-perturbative* functionals that have the following perturbative asymptotic expansion as $t \to 0$

$$u_l(t;\alpha) \sim \sum_{n\geq 0} \frac{(-1)^{n+l}}{(n+l)!} (\alpha t)^n a_{l+n,n}. \tag{2.15}$$

The zeroth order of this approximation,

$$[U(t;\alpha,\varepsilon)]\Big|_{\varepsilon=0} \sim (4\pi t)^{-d/2} u_0(t;\alpha), \tag{2.16}$$

corresponds to covariantly constant background

$$\Re_{(i)} = 0 \qquad \text{for } i \geq 1,$$

or, more explicitly,

$$\nabla_\mu R_{\alpha\beta\gamma\delta} = 0, \qquad \nabla_\mu \mathcal{R}_{\alpha\beta} = 0, \qquad \nabla_\mu Q = 0. \tag{2.17}$$

The zeroth order functional $u_0(t;\alpha)$ has the following perturbative asymptotic expansion

$$u_0(t;\alpha) \sim \sum_{n\geq 0} \frac{(-1)^n}{n!} (\alpha t)^n a_{n,n}, \tag{2.18}$$

or, symbolically,

$$u_0(t;\alpha) \sim \sum_{n\geq 0} \sum (\alpha t \Re)^n, \tag{2.19}$$

and can be viewed on as the *generating function* for that part of the HMDS-coefficients, $a_{k,k}$, that does not contain any covariant derivatives (last eq. in (2.10)).

3. Algebraic approach

There exist a very elegant indirect possibility to construct the heat kernel without solving the heat equation but using only the commutation relations of some covariant first order differential operators [22-27]. The main idea is in a generalization of the usual Fourier transform to the case of operators and consists in the following. Let us consider for a moment a trivial case of vanishing curvatures but not the potential term

$$R_{\alpha\beta\gamma\delta} = 0, \qquad \mathcal{R}_{\alpha\beta} = 0, \qquad Q \neq 0. \tag{3.1}$$

In this case the operators of covariant derivatives obviously commute and form together with the potential term an Abelian algebra

$$[\nabla_\mu, \nabla_\nu] = 0, \qquad [\nabla_\mu, Q] = 0. \tag{3.2}$$

It is easy to show that the heat *operator* can be presented in the form

$$\exp(t\,\Box) = (4\pi t)^{-d/2} \int dk g^{1/2} \exp\left(-\frac{1}{4t} k^\mu g_{\mu\nu} k^\nu\right) \exp\left(k^\mu \nabla_\mu\right), \tag{3.3}$$

where it is assumed that the covariant derivatives commute also with the metric $[\nabla_\mu, g_{\alpha\beta}] = 0$. Acting with this operator on the δ-function and using the obvious relation

$$\exp(k^\mu \nabla_\mu)\delta(x,x')\Big|_{x=x'} = \delta(k) \tag{3.4}$$

one can simply integrate over k in (3.3) to obtain the heat kernel in coordinate representation. The heat kernel diagonal is given then by

$$[U(t)] = (4\pi t)^{-d/2} \exp\left(-tQ\right). \tag{3.5}$$

In fact, the covariant differential operators ∇ do not commute and the commutators of them are proportional to the curvatures $\Re$. The commutators of covariant derivatives with the curvatures give the first derivatives of the curvatures , i.e. the jets $\Re_{(1)}$, the commutators of covariant derivatives with $\Re_{(1)}$ give the second jets $\Re_{(2)}$ etc.

$$\begin{aligned}
[\nabla, \nabla] &= \Re, \\
[\nabla, \Re] &= \Re_{(1)}, \\
&\dots \\
[\nabla, \Re_{(i)}] &= \Re_{(i+1)}, \\
&\dots
\end{aligned} \tag{3.6}$$

The commutators of jets themselves are the product of jets again

$$[\Re_{(i)}, \Re_{(k)}] = \Re_{(i+k+2)} + \sum_{0\le n\le k} \sum \Re_{(n)}\Re_{(i+k-n)}, \tag{3.7}$$

(in Abelian case all such commutators vanish).

Thus the operators of covariant differentiation ∇ together with the whole set of the jets $\mathcal{J}$ form an *infinite* dimensional Lie algebra $\mathcal{G} = \{\nabla, \Re_{(i)}\}$. To gain greater insight into how the low-energy heat kernel looks like, one can take into account a *finite* number of low-order jets, i.e. the low-order covariant derivatives of the background fields, and neglect all the higher order jets, i.e. the covariant derivatives of higher orders. Then one can show that there exist a set of covariant differential operators that together with the background fields and their low-order derivatives generate a *finite* dimensional Lie algebra $\mathcal{G}'$. This procedure is very similar to the polynomial approximation of functions of real variables. The difference is that we are dealing, in general, with the *covariant* derivatives and the curvatures.

Thus one can try to generalize the above idea in such a way that (3.3) would be the zeroth approximation in the commutators of the covariant derivatives, i.e. in the curvatures. Roughly speaking, we are going to find a representation of the heat kernel *operator* in the form

$$\exp(t\,\Box) = \int dk \Omega(t,k) \exp\left\{-\frac{1}{4t} k^A \Pi_{AB}(t) k^B\right\} \exp(k^A \xi_A), \tag{3.8}$$

where $\xi_A = \{X_a, Y_i\}$, $X_a = X_a^\mu(x)\nabla_\mu$ are some first order differential operators and $Y_i(x)$ are some functions. The functions $\Pi(t)$ and $\Omega(t,k)$ are expressed in terms of the commutators of this operators, i.e. in terms of the curvatures.

In general, the operators ξ_A do not form a closed finite dimensional Lie algebra because at each stage taking more commutators there appear more and more derivatives of the curvatures. If one restricts oneself to the low-order jets, this algebra closes and becomes finite dimensional.

Using this representation one could, as above, act with $\exp(k_A\xi^A)$ on the δ-function on M to get the heat kernel. The main point of this idea is that it is much easier to calculate the action of the exponential of the *first* order operator $k^A\xi_A$ on the δ-function than that of the exponential of the second order operator $\Box$.

4. Heat kernel in flat space

In this section we consider the more complicated case of nonvanishing covariantly constant Yang-Mills curvature in the flat space

$$R_{\alpha\beta\gamma\delta} = 0, \qquad \nabla_\mu \mathcal{R}_{\alpha\beta} = 0. \tag{4.1}$$

As we will study only *local* effects in the low-energy approximation, we will not take care about the topology of the manifold M. To be precise one can take, for example, $\mathbb{R}^d$.

4.1. Covariantly constant potential term

First we consider the case of covariantly constant potential term

$$\nabla_\mu Q = 0. \tag{4.2}$$

In this case the covariant derivatives form a *nilpotent* Lie algebra

$$\begin{aligned}
&[\nabla_\mu, \nabla_\nu] = \mathcal{R}_{\mu\nu}, \\
&[\nabla_\mu, \mathcal{R}_{\alpha\beta}] = [\nabla_\mu, Q] = [\mathcal{R}_{\mu\nu}, \mathcal{R}_{\alpha\beta}] = [\mathcal{R}_{\mu\nu}, Q] = 0.
\end{aligned} \tag{4.3}$$

For this algebra one can prove a theorem expressing the heat operator in terms of an average over the corresponding Lie group [22,23]

$$\begin{aligned}
\exp(t\,\Box) =& (4\pi t)^{-d/2} \det\left(\frac{t\mathcal{R}}{\sinh(t\mathcal{R})}\right)^{1/2} \\
&\int dk g^{1/2} \exp\left\{-\frac{1}{4t}k_\lambda (t\mathcal{R}\coth(t\mathcal{R}))^\lambda{}_\nu k^\nu\right\} \exp\left(k^\mu \nabla_\mu\right),
\end{aligned} \tag{4.4}$$

where $\mathcal{R} = \{\mathcal{R}^\mu{}_\nu\}$ means the matrix with spacetime indices and the determinant is taken with respect to these indices, other indices being intact.

It is not difficult to show that [22,23]

$$\exp(k^\mu\nabla_\mu)\mathcal{P}(x,x')\delta(x,x')\Big|_{x=x'} = \delta(k). \tag{4.5}$$

Subsequently, the integral over k^μ becomes trivial and one obtains immediately the heat kernel diagonal

$$[U(t)] = (4\pi t)^{-d/2}\det\left(\frac{t\mathcal{R}}{\sinh(t\mathcal{R})}\right)^{1/2}\exp(-tQ)\,. \tag{4.6}$$

Expanding it in a power series in t one can find *all* coefficients $a_{k,k}$ (2.10), i.e. *all* covariantly constant terms in *all* HMDS-coefficients a_k (2.7).

As we have seen the contribution of the Yang-Mills curvature is not as trivial as that of the potential term. However, the algebraic approach does work in this case too. This is the generalization of the well known Schwinger result [1] in quantum electrodynamics. It is a good example how one can get the heat kernel without solving any differential equations but using only the algebraic properties of the covariant derivatives. This result was applied for calculating the one-loop low-energy effective action in the non-Abelian gauge theory and for studying the stability of the vacuum [28].

4.2. Inclusion of the first and second derivatives of the potential term

Now we consider the case when the first and the second derivatives of the potential term do not vanish but all the higher derivatives do, i.e

$$\nabla_\mu\nabla_\nu\nabla_\lambda Q = 0. \tag{4.7}$$

Besides we assume the background to be *Abelian*, i.e. all the nonvanishing background quantities, $\mathcal{R}_{\alpha\beta}$, Q, $Q_{;\mu}$, $Q_{;\nu\mu}$, commute with each other. Thus we have a nilpotent Lie algebra $\{\nabla_\mu, \mathcal{R}_{\alpha\beta}, Q, Q_{;\mu}, Q_{;\nu\mu}\}$

$$\begin{gathered}
[\nabla_\mu,\nabla_\nu] = \mathcal{R}_{\mu\nu},\\
[\nabla_\mu, Q] = Q_{;\mu}\\
[\nabla_\mu, Q_\nu] = Q_{;\nu\mu}\\
[\mathcal{R}_{\alpha\beta},\mathcal{R}_{\mu\nu}] = [\mathcal{R}_{\alpha\beta},Q_{;\nu}] = [\mathcal{R}_{\alpha\beta},Q_{;\mu\nu}] = [Q,Q_{;\mu}] = [Q,Q_{;\mu\nu}] = [Q_{;\mu},Q_{;\alpha\beta}] = 0,
\end{gathered} \tag{4.8}$$

where $Q_{;\mu} \equiv \nabla_\mu Q$, $Q_{;\nu\mu} \equiv \nabla_\mu\nabla_\nu Q$.

For our purposes, it is helpful to introduce the following parametrization of the potential term

$$Q = M - \beta^{ik}L_iL_k, \tag{4.9}$$

where $(i = 1,\dots,p)$, $p \le d$, β^{ik} is some constant symmetric nondegenerate $p\times p$ matrix, M is a covariantly constant matrix and L_i are some matrices with vanishing *second* covariant derivative

$$\nabla_\mu M = 0, \qquad \nabla_\mu\nabla_\nu L_i = 0. \tag{4.10}$$

This gives us another nilpotent Lie algebra, $\{\nabla_\mu,\ \mathcal{R}_{\alpha\beta},\ M,\ L_i,\ L_{i;\mu}\}$, with following nontrivial commutators

$$[\nabla_\mu, \nabla_\nu] = \mathcal{R}_{\mu\nu}, \qquad [\nabla_\mu, L_i] = L_{i;\mu}, \tag{4.11}$$

and the center $\{\mathcal{R}_{\alpha\beta},\ M,\ L_i,\ L_{i;\mu}\}$. Introducing the generators $\xi_A = (\nabla_\mu, L_i)$, $(A = 1, \dots, D)$, $D = d+p$, one can rewrite these commutation relations in a more compact form

$$[\xi_A, \xi_B] = \mathcal{F}_{AB}, \tag{4.12}$$

$$[\xi_A, \mathcal{F}_{CD}] = [\mathcal{F}_{AB}, \mathcal{F}_{CD}] = 0,$$

where $\mathcal{F}_{AB}$ is a matrix

$$(\mathcal{F}_{AB}) = \begin{pmatrix} \mathcal{R}_{\mu\nu} & L_{i;\mu} \\ -L_{k;\nu} & 0 \end{pmatrix}, \tag{4.13}$$

that we call the *generalized* curvature. The operator H (1.4) can now be written in the form

$$H = -\gamma^{AB}\xi_A\xi_B + M, \tag{4.14}$$

where

$$(\gamma^{AB}) = \begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \beta^{ik} \end{pmatrix}. \tag{4.15}$$

The matrices β^{ik} and γ^{AB} play the role of metrics and can be used to raise and to lower the small and the capital Latin indices respectively.

Note that the algebra (2.10) is essentially of the same type as (4.3). For algebras of this kind the heat operator is given by the integral over the corresponding Lie group [22,23]

$$\begin{aligned} \exp(-tH) =& (4\pi t)^{-D/2} \det\left(\frac{\sinh(t\mathcal{F})}{t\mathcal{F}}\right)^{-1/2} \exp(-tM) \\ & \times \int\limits_{\mathbb{R}^D} dk \gamma^{1/2} \exp\left\{-\frac{1}{4t} k_A (t\mathcal{F}\coth(t\mathcal{F}))^A{}_B k^B\right\} \exp(k^A\xi_A), \end{aligned} \tag{4.16}$$

where $\gamma = \det \gamma_{AB}$.

Thus we have expressed the heat kernel operator in terms of the operator $\exp(k^A\xi_A)$. The integration over k in (4.16) is Gaussian except for noncommutative part. Splitting the integration variables $(k^A) = (q^\mu, \omega^i)$ and using the Campbell-Hausdorf formula we obtain [27]

$$\exp(k^A\xi_A)\delta(x, x')\Big|_{x=x'} = \exp(\omega^i L_i)\delta(q), \tag{4.17}$$

hence taking off the integration over q. After integrating over ω we obtain the heat kernel diagonal in a very simple form [27]

$$[U(t)] = (4\pi t)^{-d/2}\Phi(t) \exp\left\{-tQ + \tfrac{1}{4}t^3 Q_{;\mu}\Psi^{\mu\nu}(t) Q_{;\nu}\right\}. \tag{4.18}$$

where

$$\Phi(t) = \det\left(\frac{\sinh(t\mathcal{F})}{t\mathcal{F}}\right)^{-1/2} \det(1 + t^2 C(t) P)^{-1/2}, \tag{4.19}$$

$$\Psi(t) = (\Psi^\mu{}_\nu(t)) = (1 + t^2 C(t) P)^{-1} C(t), \tag{4.20}$$

P is the matrix of second derivatives of the potential term,

$$P = \left\{\tfrac{1}{2} Q^{;\mu}{}_{;\nu}\right\} \tag{4.21}$$

and the matrix $C(t) = \{C^\mu{}_\nu(t)\}$ is defined by

$$C(t) = \oint\limits_C \frac{dz}{2\pi i} t\coth(tz^{-1})(1 - z\mathcal{R} - z^2 P)^{-1}. \tag{4.22}$$

The formula (4.18) exhibits the general structure of the heat kernel diagonal. Namely, one sees immediately how the potential term and its first derivatives enter the result. The complete nontrivial information is contained only in a scalar, $\Phi(t)$, and a tensor, $\Psi_{\mu\nu}(t)$, functions which are constructed purely from the Yang-Mills curvature $\mathcal{R}_{\mu\nu}$ and the *second* derivatives of the potential term, $\nabla_\mu \nabla_\nu Q$. So we conclude that the coefficients of the heat kernel asymptotic expansion are constructed from three different types of scalar (connected) blocks, Q, $\Phi_{(n)}(\mathcal{R}, \nabla\nabla Q)$ and $\nabla_\mu Q \Psi^{\mu\nu}_{(n)}(\mathcal{R}, \nabla\nabla Q) \nabla_\nu Q$.

In a special case, when the matrices $\mathcal{R}$ and P commute, i.e.

$$[\mathcal{R}, P] = 0, \qquad \text{or} \qquad \mathcal{R}^\mu{}_\nu P^\nu{}_\alpha = P^\mu{}_\nu \mathcal{R}^\nu{}_\alpha, \tag{4.23}$$

the heat kernel diagonal simplifies considerably [27]

$$\begin{aligned}[U(t)] = (4\pi t)^{-d/2} &\det\left(\frac{\sinh(t\Delta)}{t\Delta}\right)^{-1/2} \\ &\times \exp\left\{-tQ + \tfrac{1}{4} t Q_{;\mu}\left[\frac{1}{P}\left(\frac{\Delta}{2tP}\frac{\cosh(t\mathcal{R}) - \cosh(t\Delta)}{\sinh(t\Delta)} + 1\right)\right]^\mu{}_\nu Q^{;\nu}\right\},\end{aligned} \tag{4.24}$$

where

$$\Delta = \sqrt{\mathcal{R}^2 + 4P}. \tag{4.25}$$

If the second derivatives of the potential vanish, $P_{\mu\nu} = \frac{1}{2}\nabla_\mu \nabla_\nu Q = 0$, then we have therefrom

$$\begin{aligned}[U(t)] =&(4\pi t)^{-d/2} \det\left(\frac{\sinh(t\mathcal{R})}{t\mathcal{R}}\right)^{-1/2} \\ &\times \exp\left\{-tQ + \tfrac{1}{4} t Q_{;\mu}\left(\frac{1}{\mathcal{R}^2}(t\mathcal{R}\coth(t\mathcal{R}) - 1)\right)^\mu{}_\nu Q^{;\nu}\right\}.\end{aligned} \tag{4.26}$$

This is the first order correction to the case of covariantly constant potential [22] when additionally the *first* derivatives of the potential are taken into account.

In the case of vanishing Yang-Mills curvature, $\mathcal{R}=0$, we have similarly

$$\begin{aligned}[U(t)] =&(4\pi t)^{-d/2}\det\left(\frac{\sinh(2t\sqrt{P})}{2t\sqrt{P}}\right)^{-1/2}\\ &\times\exp\left\{-tQ-\tfrac{1}{4}Q_{;\mu}\left(\frac{\tanh(t\sqrt{P})-t\sqrt{P}}{P^{3/2}}\right)^{\mu}{}_{\nu}Q^{;\nu}\right\}.\end{aligned} \tag{4.27}$$

This determines the low-energy approximation without the gauge fields.

4.3. Trace of the heat kernel

Let us now calculate the trace of the heat kernel. We assumed that the background fields satisfy the low-energy conditions (4.7) in some region of the manifold M. Let us suppose the manifold M to be $\mathbb{R}^d$ and the conditions (4.7) to hold everywhere. Then the formula for the heat kernel diagonal (4.18) is valid everywhere too. Let the matrix P (4.21) to be nondegenerate, then one can integrate (4.18) over $\mathbb{R}^d$ to get [27]

$$\mathrm{Tr}\, U(t) = (2t)^{-d}\det\left(\frac{\sinh(t\mathcal{F})}{t\mathcal{F}}\right)^{-1/2}\det P^{-1/2}\exp(-tM). \tag{4.28}$$

In particular case of commuting matrices $\mathcal{R}$ and P the trace of the heat kernel takes especially simple form

$$\mathrm{Tr}\, U(t) = \det\left\{2(\cosh(t\Delta)-\cosh(t\mathcal{R}))\right\}^{-1/2}\exp(-tM), \tag{4.29}$$

which reduces to

$$\mathrm{Tr}\, U(t) = \det\left(2\sinh(t\sqrt{P})\right)^{-1}\exp(-tM), \tag{4.30}$$

when $\mathcal{R}=0$.

It should be noted that these expressions have *nonclassic* asymptotics, $\mathrm{Tr}\, U(t) \sim \mathrm{const}\cdot t^{-d}$ instead of the usual standard one $\mathrm{Tr}\, U(t)\sim \mathrm{const}\cdot t^{-d/2}$ that holds on compact manifolds. The standard form of the asymptotics of the trace of the heat kernel like (2.11) is the basis for the regularization and renormalization procedure in quantum field theory [2]. That is why the non-standard asymptotics may cause serious technical problems in the theory of quantum fields on noncompact manifolds with background fields that do not fall off at infinity. For example, the analytical structure of the zeta function (1.2) in non-standard case will be completely different. This is the consequence of the fact that in this non-standard situation the physical quantum states are not well defined.

5. Heat kernel in symmetric spaces

Let us now try to generalize the algebraic approach to the case of the *curved* manifolds with covariantly constant Riemann curvature, covariantly constant potential and vanishing Yang-Mills curvature,

$$\nabla_\mu R_{\alpha\beta\gamma\delta}=0, \qquad \mathcal{R}_{\mu\nu}=0, \qquad \nabla_\mu Q=0. \tag{5.1}$$

A complete simply connected Riemannian manifold with covariantly constant curvature is called symmetric space. A symmetric space is said to be of compact, noncompact or Euclidean type if all sectional curvatures $K(u,v) = R_{abcd}u^a v^b u^c v^d$ are positive, negative or zero. A direct product of symmetric spaces of compact and noncompact types is called *semisimple* symmetric space. It is well known that a generic complete simply connected Riemannian symmetric space is a direct product of a flat space and a semisimple symmetric space [35,36].

It should be repeated here once more that our analysis in this paper is purely *local*. We are looking for a *universal local* function of the curvature, $u_0(t)$, (2.16) that describes adequately the low-energy limit of the heat kernel diagonal. Our minimal requirement is that this function should reproduce *all* the terms without covariant derivatives of the curvature in the local Schwinger-De Witt asymptotic expansion of the heat kernel (2.6), (2.18), i.e. it should give *all* the HMDS-coefficients a_k (2.7) for *any* symmetric space.

Since the HMDS-coefficients have a *universal* explicit structure [7], it is obvious that any flat subspaces do not contribute in a_k. Moreover, since HMDS-coefficients a_k are analytic in the curvature it is evident that to find this universal structure it is sufficient to consider only symmetric spaces of *compact* type with positive curvature. Using the factorization property of the heat kernel [7] and the duality between compact and noncompact symmetric spaces [35,36] one can obtain then the results for the general case by analytical continuation. That is why we consider only the case of symmetric spaces of *compact* type.

First of all, we choose a frame $e^\mu_a(x,x')$ that is *covariantly constant (parallel)* along the geodesic between the points x and x'. Let us consider the Riemann tensor in more detail. It is obvious that the frame components of the curvature tensor of a symmetric space are constant. For *any* Riemannian manifold they can be presented in the form

$$R_{abcd} = \beta_{ik} E^i{}_{ab} E^k{}_{cd}, \tag{5.2}$$

where E^i_{ab}, $(i = 1,\dots,p; p \le d(d-1)/2)$, is some set of antisymmetric matrices and β_{ik} is some symmetric nondegenerate $p\times p$ matrix. The traceless matrices $D_i = \{D^a{}_{ib}\}$ defined by

$$D^a{}_{ib} = -\beta_{ik} E^k{}_{cb} g^{ca} = -D^a{}_{bi} \tag{5.3}$$

are known to be the generators of the *holonomy algebra* $\mathcal{H}$ [36]

$$[D_i, D_k] = F^j{}_{ik} D_j, \tag{5.4}$$

where $F^j{}_{ik}$ are the structure constants.

In symmetric spaces there exists a much wider Lie algebra $\mathcal{G}$ of dimension $D = p + d$ [24-26]

$$[C_A, C_B] = C^C{}_{AB} C_C, \tag{5.5}$$

where the structure constants $C^A{}_{BC}$, $(A = 1,\dots,D)$ are defined by

$$C^i{}_{ab} = E^i{}_{ab}, \quad C^a{}_{ib} = D^a{}_{ib}, \quad C^i{}_{kl} = F^i{}_{kl}, \tag{5.6}$$

$$C^a{}_{bc} = C^i{}_{ka} = C^a{}_{ik} = 0,$$

and $C_A = \{C^B{}_{AC}\}$ are the generators of adjoint representation. Thus the structure of the algebra $\mathcal{G}$ is completely determined by the curvature tensor of symmetric space.

Moreover, in case of semisimple symmetric space the algebra $\mathcal{G}$ is isomorphic to the algebra of the infinitesimal isometries, i.e. the Killing vector fields ξ_A, [35,36]

$$[\xi_A, \xi_B] = C^C{}_{AB}\xi_C. \tag{5.7}$$

Therefore, the curvature tensor of the semisimple symmetric space completely determines the structure of the group of isometries too.

In semisimple symmetric spaces the Laplacian can be presented in terms of generators of isometries [24-26]

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu = \gamma^{AB}\xi_A\xi_B, \tag{5.8}$$

where

$$\gamma^{AB} = \begin{pmatrix} g^{ab} & 0 \\ 0 & \beta^{ik} \end{pmatrix} \tag{5.9}$$

and $\beta^{ik} = (\beta_{ik})^{-1}$.

Using this representation one can prove a theorem expressing the heat operator in terms of some average over the group of isometries G [24-26]

$$\begin{aligned} \exp(t\,\Box) =& (4\pi t)^{-D/2}\int dk\gamma^{1/2}\det\left(\frac{\sinh\left(k^A C_A/2\right)}{k^A C_A/2}\right)^{1/2} \\ & \times \exp\left\{-\frac{1}{4t}k^A\gamma_{AB}k^B + \frac{1}{6}R_G t\right\}\exp(k^A\xi_A) \end{aligned} \tag{5.10}$$

where $\gamma = \det\gamma_{AB}$, $\gamma_{AB} = (\gamma^{AB})^{-1}$ and R_G is the scalar curvature of the group of isometries G

$$R_G = -\frac{1}{4}\gamma^{AB}C^C{}_{AD}C^D{}_{BC}. \tag{5.11}$$

The proof of this theorem is given in [24,25].

Splitting the integration variables $k^A = (q^a, \omega^i)$ one can find first the action of the isometries on the δ-function [24-26]

$$\exp\left(k^A\xi_A\right)g^{-1/2}\delta(x,x')\Big|_{x=x'} = \det\left(\frac{\sinh\left(\omega^i D_i/2\right)}{\omega^i D_i/2}\right)^{-1}\eta^{-1/2}\delta(q), \tag{5.12}$$

where $\eta = \det g_{ab}$. Then one can easily integrate over q in (5.10) to get heat kernel diagonal [24-26]

$$\begin{aligned} [U(t)] =& (4\pi t)^{-D/2}\int d\omega\beta^{1/2}\det\left(\frac{\sinh\left(\omega^i F_i/2\right)}{\omega^i F_i/2}\right)^{1/2}\det\left(\frac{\sinh\left(\omega^i D_i/2\right)}{\omega^i D_i/2}\right)^{-1/2} \\ & \times \exp\left\{-\frac{1}{4t}\omega^i\beta_{ik}\omega^k - \left(Q - \frac{1}{8}R - \frac{1}{6}R_H\right)t\right\}, \end{aligned} \tag{5.13}$$

where $\beta = \det\beta_{ik}$, $F_i = \{F^k{}_{ij}\}$ are the generators of the holonomy algebra in adjoint representation and R_H is the scalar curvature of the holonomy group H

$$R_H = -\frac{1}{4}\beta^{ik}F^m{}_{il}F^l{}_{km}. \tag{5.14}$$

One can present this result also in an alternative nontrivial rather *formal* way without any integration [24-26]

$$\begin{aligned}[U(t)](t) = (4\pi t)^{-d/2}\exp\left\{\left(\frac{1}{8}R + \frac{1}{6}R_H - Q\right)t\right\}\det\left(\frac{\sinh(\sqrt{t}\partial^k F_k/2)}{\sqrt{t}\partial^k F_k/2}\right)^{1/2} \\ \times\det\left(\frac{\sinh(\sqrt{t}\partial^k D_k/2)}{\sqrt{t}\partial^k D_k/2}\right)^{-1/2}\exp\left(p_n\beta^{nk}p_k\right)\Bigg|_{p=0}.\end{aligned} \tag{5.15}$$

where p_i are some auxiliary variables and $\partial^k = \partial/\partial p_k$. This formal solution should be understood as a power series in the derivatives ∂^i that is well defined and determines the heat kernel asymptotic expansion at $t \to 0$, i.e. *all* HMDS-coefficients a_k.

Let us stress that the closed formulae obtained in this section are *exact* (up to possible nonanalytic topological contributions) and *manifestly covariant* because they are expressed in terms of the invariants of the holonomy group H, i.e. the invariants of the curvature tensor. They can be used now to generate *all* HMDS-coefficients a_k for *any* symmetric space, i.e. for *any space with covariantly constant curvature*, simply by expanding it in a power series in t. Thereby one finds *all covariantly constant terms in all HMDS-coefficients* in a manifestly covariant way. This gives a very nontrivial example how the heat kernel can be constructed using only the commutation relations of some differential operators, namely the generators of infinitesimal isometries of the symmetric space. We are going to obtain the explicit formulae in a further work.

Although we considered for simplicity the case of symmetric space of *compact* type, i.e. with positive sectional curvatures, i.e. positive definite matrix β_{ik}, it is not difficult to generalize our results to the general case using the duality relation and the *analytic continuation*. This means that our formulae for the asymptotic expansion of the heat kernel should be valid in general case of arbitrary symmetric space too. Moreover, they do not depend on the signature of the spacetime metric and should also be valid for the case of Lorentzian signature.

6. Conclusion

We have presented a brief overview of recent results in studying the heat kernel obtained in our papers [22-28]. We discussed some ideas connected with the structure of the asymptotic expansions of the heat kernel with respect to some deformation parameters. These asymptotic expansions allow to develop a new scheme for covariant calculation of the heat kernel in low-energy approximation and to calculate explicitly the heat kernel

diagonal in zeroth order. The main idea of this approximation scheme is to employ the low-energy background jet algebra.

We obtained closed formulas for the heat kernel diagonal that can be treated as a resummation of the asymptotic expansion, the covariantly constant terms (and some low-order derivatives terms), being summed up first. The covariant algebraic approach is especially adequate and effective to study the low-energy approximation. It seems that it can be developed deeper and that it can be formulated a general technique for systematic calculation of the low-energy heat kernel, a kind of *low-energy covariant perturbation theory.*

Among unsolved problems one should note, first of all, the problem of generalizing our results to the most general covariantly constant background including the Yang-Mills curvature. Then, it is very interesting to obtain *explicitly* the covariantly constant terms in HMDS-coefficients, i.e. to calculate $a_{k,k}$ part of HMDS-coefficients. This would be the opposite case to the higher-derivative approximation and can be of certain interest in mathematical physics. Finally, it is not perfectly clear how to do the analytical continuation to the spacetime of Lorentzian signature. All the activity in calculating the low-energy heat kernel is motivated by the physical problem of studying the vacuum structure in quantum gravity and gauge theories.

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